Douglas College

Douglas College Learning Centre

QUADRATIC EQUATIONS AND FUNCTIONS

Quadratic equations and functions are very important in Business Math. Questions related to quadratic equations and functions cover a wide range of business concepts including cost/revenue, break-even analysis, supply/demand, market equilibrium, and so on. This handout explains concepts and provides exercises for solving these types of questions.

The handout has three parts:

I. Quadratic Equations
   The focus of this part is to master the techniques for solving quadratic equations using the quadratic formula and factoring.

II. Quadratic Functions
   The focus of this part is to learn the properties of quadratic functions and to graph parabolas.

III. Applied Studies
   The focus of this part is to incorporate the techniques from the previous two parts and to solve various types of business related questions (word problems).

I. Quadratic Equations

$ax^2 + bx + c = 0$ ($a \neq 0$)

Special Forms

$x^2 = n$ if $n < 0$, then x has no real value

Example: Solve x.
$x^2 = -5$ x has no

$x^2 = n$ if $n > 0$, then $x = \pm\sqrt{n}$

Example: Solve x.
$x^2 = 16$ $x = \pm 4$

$ax^2 + bx = 0$ $x = 0$, $x = -b/a$

Example: Solve x.
$x^2 - 3x = 0$
$x(x - 3) = 0$ $x = 0$, $x = 3$

Quadratic Formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

As we can see from the formula, $b^2 - 4ac$ is under the square root sign.

If $b^2 - 4ac < 0$, x has no real value;

If $b^2 - 4ac = 0$, x has one value: $x = \frac{-b}{2a}$

If $b^2 - 4ac > 0$, x has two values: $x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$

$$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Factoring

Factoring means to rewrite the quadratic equation into multiplication form.

For example, quadratic equation

$x^2 - 7x + 6 = 0$

can be rewritten in the form of

$(x – 1)(x – 6) = 0$

Factoring is a very useful way for solving quadratic equations. Using the example above, solve x in the factored quadratic equation

$$(x - 1)(x - 6) = 0$$

x- 1 = *x- 6 = 0* → Answer: <u>x = 1</u>, <u>x = 6</u>

All valid quadratic equations can be written in factoring form, so factoring can appear as an independent form of question.

There are several formulas useful for factoring a quadratic equation:

1. $x^2 + 2ax + a^2 = 0$ → $(x + a)^2 = 0$

> Example: Factor the following
>
> $x^2 + 10x + 25 = 0$ *Analyze:* $(x^2 + 2 \cdot 5x + 5^2) = 0$
>
> Answer: <u>$(x + 5)^2 = 0$</u>

2. $x^2 - 2ax + a^2 = 0$ → $(x - a)^2 = 0$

   $x^2 - a^2 = 0$ → $(x + a)(x – a) = 0$

> Example: Write the following equation in factoring form.
>
> $x^2 - 36 = 0$ *Analyze:* $(x^2 - 6^2) = 0$
>
> Answer: <u>$(x + 6)(x – 6) = 0$</u>

Special Formula for Factoring

Formula	Example
$ax^2 + bx + c = 0$ a_1 ╲╱ c_1 a_2 ╱╲ c_2 $a_1 \cdot a_2 = a$ $c_1 \cdot c_2 = c$ $a_1 \cdot c_2 + a_2 \cdot c_1 = b$ Answer: $(a_1x + c_1)(a_1x + c_2) = 0$	$6x^2 - x - 2 = 0$ *a=6, b=-1, c=-2* 2 ╲╱ 1 3 ╱╲ -2 $a_1 \cdot a_2 = 2 \cdot 3 = a$ $c_1 \cdot c_2 = 1 \cdot (-2) = -2 = c$ $a_1 \cdot c_2 + a_2 \cdot c_1 = 2 \cdot (-2) + 3 \cdot 1 = -1 = b$ Answer: $(2x + 1)(3x - 2) = 0$

Quadratic Equations Exercise

1. Solve x (hint: by using special forms of quadratic equations)

a) $x^2 = -1$

b) $x^2 - 1 = 0$

c) $x^2 - 7x = 0$

d) $x^2 = -3x$

2. Solve x (hint: quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$)

a) $3x^2 - x + 2 = 0$

b) $x^2 - x + 1 = 0$

c) $-x^2 + 9x - 18 = 0$

d) $4x^2 + 8x - 5 = 0$

3. Rewrite the following quadratic equations in factoring forms.

a) $x^2 - x - 20 = 0$

b) $2x^2 + 3x - 5 = 0$

4. Solve x (hint: by factoring)

a) $x^2 - x - 12 = 0$

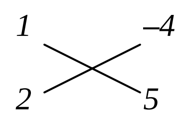
b) $3x^2 + 6x - 9 = 0$

Answers for Quadratic Equations Exercise:

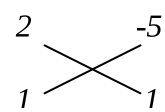
1. a) no valid answer ($b^2 - 4ac < 0$)
   b) $x = \pm 1$

2. a) no valid answer ($b^2 - 4ac < 0$)
   b) no valid answer ($b^2 - 4ac < 0$)
   c) 3, -6
   d) ½, 2½

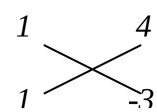
3. a) $(x - 5)(x + 4) = 0$



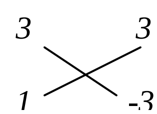
   b) $(2x + 5)(x - 1) = 0$



4. a) –3, 4 $(x - 4)(x + 3) = 0$



   b) –3, 1 $(3x - 3)(x + 3) = 0$



II. Quadratic Functions

$f(x) = ax^2 + bx + c \ (a \neq 0)$

Properties

The graph of a quadratic function is called a parabola. In business math, we consider only some of the simple properties of the quadratic functions and their graphing.

Properties necessary to remember:

1. the parabola is symmetrical to a vertical line: $x = -b/2a$.
2. when $a > 0$, the parabola opens upward, the corresponding quadratic function has a minimum value $f(-b/2a)$.
3. when $a < 0$, the parabola opens downward, the corresponding quadratic function has a maximum value $f(-b/2a)$.
4. the values of x for the quadratic equation $ax^2 + bx + c = 0$ correspond to the intercepts of the parabola and the x-axis. Thus

   - when $b^2 - 4ac > 0$, $ax^2 + bx + c = 0$ has two x values, the parabola of the function $f(x) = ax^2 + bx + c$ has two intercepts with the x-axis.

   - when $b^2 - 4ac = 0$, $ax^2 + bx + c = 0$ has one x value, the parabola of the function $f(x) = ax^2 + bx + c$ has one intercept with the x-axis, and this intercept is the minimum/maximum value of the function.

   - when $b^2 - 4ac > 0$, $ax^2 + bx + c = 0$ has no x value, the parabola of the function $f(x) = ax^2 + bx + c$ has no intercept with the x-axis.

5. Vertex (-b/2a, f(-b/2a) is the lowest/highest point on a parabola. This point can help decide the positioning of the parabola, and also helps to solve the minimum/maximum value of the quadratic function.

Graphing

For regular quadratic function graphing, you need to show at least the vertex of the parabola unless otherwise specified. For business math graphing, you need to show other points such as break-even, equilibrium, and so on.

The following example shows the standard graphing of a regular quadratic function.

Example: Graph the quadratic function: $y = x^2 - 4x + 3$

Analyze:

1. Decide the opening direction of the parabola.

   For this function, a = 1, which is > 0, so the parabola opens upwards.

2. Find vertex of this function:
   $x_v = -b/2a = 4/2 = 2$
   $y_v = (2)^2 - 4(2) + 3 = -1$

   Vertex (2, -1)

3. Decide two other points
   (There are many different ways to find at least two other points on the parabola. This example finds the parabola's two intercepts with x-axis, you can also plug x values into the function to find other points.)

   x-intercepts:

   $x^2 - 4x + 3 = 0$ $x = 1, x = 3$

   so points (1, 0) and (3, 0) are on the parabola.

4. Decide scales.

5. Graph.

Quadratic Functions Exercise

Graph the following quadratic functions:

a) $y = x^2 - 3x + 1$

b) $y = -2x^2 + 4x + 7$

Answers for Quadratic Functions Exercise:

a)

1. a=1 ➔ parabola opens upward
2. $V_{(x)} = -b/2a = 3/2 = \underline{1.5}$
   $V_{(y)} = (1.5)^2 - 3.(1.5) + 1 = \underline{-1.25}$
3. x intercepts:
   $x = \underline{2.62}$
   $x = \underline{0.38}$
4. Decide a scale
5. Graph

b)

1. a=-2 ➔ parabola opens downward
2. $V_{(x)} = -b/2a = -4/-4 = \underline{1}$
   $V_{(y)} = -2(1)^2 + 4 \cdot 1 + 7 = \underline{9}$
3. x intercepts:
   $x = \underline{-1.12}$
   $x = \underline{3.12}$

III. Applied Studies

o Cost, Revenue, and Profit

Revenue Function: R(x) = (unit price) · x
Cost Function: C(x) = (variable cost) · x + (fixed cost)
Profit Function: P(x) = R(x) – C(x)

Common types of questions relating to cost/revenue/profit include:

Find the Break-Even point

At Break-Even (there is no profit, the costs equal the revenue).
R(x) = C(x)

Example: If the total costs are $C(x) = 500 + 90x$, and total revenues are $R(x) = 150x – x^2$. Find the break-even point(s).

$R(x) = C(x)$
$150x - x^2 = 500 + 90x$
$x^2 - 60x + 500 = 0$
$x = 50$, $x = 10$

(can bring this x value to either C(x) or R(x))
$y = 500 + 90 \cdot 50 = 5000$
$y = 500 + 90 \cdot 10 = 1400$

the break-even points are: (50, 5000), (10,1400)

Generate Profit Function and find Maximum Profit

Profit Function: P(x) = R(x) – C(x)

Example: Using the above example, write the profit function, and find 1) what level production maximizes the profit? 2) what is the maximum profit?

$P(x) = R(x) – C(x)$
$= (150x – x^2) – (500 + 90x)$
$= 150x – x^2 - 500 – 90x$
$= -x^2 + 60x -500$

(at vertex, function reaches maximum point)

$V_{(x)} = -b/2a = -60/(2\cdot(-1)) = 30$
1) producing 30 units maximizes the profit.

$V_{(y)} = -30^2 + 60 \cdot 30 – 500 = 2200$
2) the maximum profit is $2,200.

Demand and Supply

Common types of questions relating to demand/supply include:

Find Market Equilibrium Point

At Market Equilibrium, Demand = Supply.

Example: if the demand function for a product is given by $p^2+2q = 1600$, and the supply function is given by $200 - p^2 + 2q = 0$, find the equilibrium quantity and equilibrium price.

First of all, rewrite the demand and supply functions.

Demand Function: $q = -(½)p^2 + 800$
Supply Function: $q = (½)p^2 - 100$
(p represents price, q represents quantity)

At Market Equilibrium, Demand = Supply

$-(½)p^2 + 800 = (½)p^2 - 100$

$p^2 - 900 = 0$

$p = 30$ ($p = -30$ is not a valid answer)

<u>the equilibrium price is $30</u>.

(can bring this price to either supply or demand function)

$q = (½)(30)^2 - 100 = 350$

<u>the equilibrium quantity is 350 units</u>.

Applied Studies Exercise

a) For producing a certain product, if total costs can be represented by $C(x) = 1600 + 1500x$, and the total revenues can be represented by $R(x) = 1600x - x^2$, find the break-even point(s) and the maximum possible profit.

b) If the demand function for a commodity is given by the equation $p^2 + 4q = 1600$, and the supply function is given by the equation $550 - p^2 + 2q = 0$, find the equilibrium quantity and equilibrium price.

Answers for Applied Studies Exercise

a) at break-even points, $C(x) = R(x)$

$1600 + 1500x = 1600x - x^2$
$x^2 - 100x + 1600 = 0$
$x = 20, x = 80$

$C(20) = 1600 + 1500\ 20 = 31600$
$C(80) = 1600 + 1500\ 80 = 121600$

The break-even points are (20, 31600), (80, 121600)

b) Demand function: $q = -(1/4)p^2 + 400$
Supply function: $q = (1/2)p^2 - 275$

at market equilibrium point, supply equals demand.

$-(1/4)p^2 + 400 = (1/2)p^2 - 275$
$-(3/4)p^2 = -675$
$p^2 = 900$
$p = 30, p = -30$ (not valid)

$q = (1/2) \cdot (30)^2 - 275 = 175$

the equilibrium price is \$30, the equilibrium quantity is 175.